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Representations of a q-analogue of the $*$-algebra Pol($\mathrm{Mat}_{2,2}$)

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Abstract

Bounded Hilbert space $*$-representations are studied for a q-analogue of the $*$-algebra $\mathrm{Pol}(\mathrm{Mat}_{2,2})$ of polynomials on the space $\mathrm{Mat}_{2,2}$ of complex 2×2 matrices.

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1. Introduction

The study of q-analogues of the Cartan domains (irreducible bounded symmetric domains) was initiated by Sinel'shchikov and Vaksman in [SV]. In particular, for each Cartan domain they defined the $*$-algebra $\mathrm{Pol}(\mathfrak{g}_{-1})_q$, a q-analogue of the polynomial algebra on the prehomogeneous vector space $\mathfrak{g}_{-1}$, and set a problem on investigation of their representations. The theory of representations of the $*$-algebras corresponding to domains of rank 1 is well understood. In this paper our purpose is to study such representations for one of the popular Cartan domains of rank 2, the matrix ball in the space $\mathrm{Mat}_{2,2}$ of complex 2×2 matrices. Following [SSV] we will denote this $*$-algebra by $\mathrm{Pol}(\mathrm{Mat}_{2,2})_q$. A description of $\mathrm{Pol}(\mathrm{Mat}_{m,n})_q$, $m, n \in \mathbb{N}$, in terms of generators and relations is given in [SSV]. In this paper we classify all irreducible representations of $\mathrm{Pol}(\mathrm{Mat}_{2,2})_q$ by bounded operators on a Hilbert space. The method which we use here is based on the study of a dynamical system arising on a spectrum of a commutative $*$-subalgebra of $\mathrm{Pol}(\mathrm{Mat}_{2,2})_q$ (see [OS]). Note that the $*$-algebra also has unbounded $*$-representation. One can easily define a 'well behaved' class of such unbounded representations and classify them up to unitary equivalence using the same technique. Finally we determine those representations of $\mathrm{Pol}(\mathrm{Mat}_{2,2})_q$ which are induced by representations of the $*$-algebra $\mathrm{Pol}(S(\mathbb{U}))_q$, a q-analogue of the polynomial algebra on the Shilov boundary. The last algebra was introduced in [V].

In this paper we use the following standard notations: $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^+$ is the set of non-negative real numbers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$, $\mathbb{N} = \{1, 2, \ldots\}$.

2. The $*$-algebra $\mathrm{Pol}(\mathrm{Mat}_{2,2})_q$ and its $*$-representations

Let $q \in (0, 1)$. The $*$-algebra $\mathrm{Pol}(\mathrm{Mat}_{2,2})_q$, a q-analogue of polynomials on the space $\mathrm{Mat}_{2,2}$ of complex 2×2 matrices, is given by its generators $\{z_a^\alpha\}_{a=1,2;\alpha=1,2}$ and the following commutation relations:

$$\begin{array}{ll} z_1^1 z_2^1 = q z_2^1 z_1^1 & z_2^1 z_1^2 = z_1^2 z_2^1 \\ z_1^1 z_1^2 = q z_1^2 z_1^1 & z_2^1 z_2^2 = q z_2^2 z_2^1 \\ z_1^1 z_2^2 - z_2^2 z_1^1 = (q - q^{-1}) z_1^2 z_2^1 & z_1^2 z_2^2 = q z_2^2 z_1^2 \end{array} \tag{1}$$

$$\begin{array}{l} (z_1^1)^* z_1^1 = q^2 z_1^1 (z_1^1)^* - (1-q^2)(z_2^1 (z_2^1)^* + z_1^2 (z_1^2)^*) + q^{-2}(1-q^2)^2 z_2^2 (z_2^2)^* + 1 - q^2 \\ (z_2^1)^* z_2^1 = q^2 z_2^1 (z_2^1)^* - (1-q^2) z_2^2 (z_2^2)^* + 1 - q^2 \\ (z_1^2)^* z_1^2 = q^2 z_1^2 (z_1^2)^* - (1-q^2) z_2^2 (z_2^2)^* + 1 - q^2 \\ (z_2^2)^* z_2^2 = q^2 z_2^2 (z_2^2)^* + 1 - q^2 \end{array} \tag{2}$$

and

$$\begin{array}{ll} (z_1^1)^* z_2^1 - q z_2^1 (z_1^1)^* = (q - q^{-1}) z_2^2 (z_1^2)^* & (z_2^2)^* z_2^1 = q z_2^1 (z_2^2)^* \\ (z_1^1)^* z_1^2 - q z_1^2 (z_1^1)^* = (q - q^{-1}) z_2^2 (z_2^1)^* & (z_2^2)^* z_1^2 = q z_1^2 (z_2^2)^* \\ (z_1^1)^* z_2^2 = z_2^2 (z_1^1)^* \qquad (z_2^1)^* z_1^2 = z_1^2 (z_2^1)^*. & \end{array} \tag{3}$$

An algebra which is generated by $\{z_a^\alpha\}_{a=1,2;\alpha=1,2}$ and relations (1) will be denoted by $\mathbb{C}(\mathrm{Mat}_{2,2})$. This algebra is a q-analogue of the algebra of holomorphic polynomials in the vector space $\mathrm{Mat}_{2,2}$.

Consider a representation π of $\mathrm{Pol}(\mathrm{Mat}_{2,2})_q$ on a separable Hilbert space H by bounded operators. The theorem below gives the complete classification of such irreducible representations up to unitary equivalence.

Theorem 1. *Any irreducible representation π is unitarily equivalent to one of the following six series:*

(1) one-dimensional representations $\xi_{\varphi_1,\varphi_2}$

$$\begin{array}{l} \xi_{\varphi_1,\varphi_2}(z_1^1) = q^{-1} \mathrm{e}^{\mathrm{i}\varphi_1} \quad \xi_{\varphi_1,\varphi_2}(z_2^1) = \xi_{\varphi_1,\varphi_2}(z_1^2) = 0 \quad \xi_{\varphi_1,\varphi_2}(z_2^2) = \mathrm{e}^{\mathrm{i}\varphi_2} \\ \varphi_i \in [0, 2\pi) \end{array} \tag{4}$$

(2) infinite-dimensional representations π_φ on $H = l_2(\mathbb{Z}^+)$

$$\begin{array}{l} \pi_\varphi(z_1^1) e_k = q^{-1} \sqrt{1 - q^{2(k+1)}}\, e_{k+1} \\ \pi_\varphi(z_2^2) e_k = \mathrm{e}^{\mathrm{i}\varphi} e_k \qquad \pi_\varphi(z_2^1) = \pi_\varphi(z_1^2) = 0 \\ \varphi \in [0, 2\pi) \end{array} \tag{5}$$

(3) infinite-dimensional representations $\rho_{\varphi_1,\varphi_2}$ on $H = l_2(\mathbb{Z}^+)$

$$\begin{array}{l} \rho_{\varphi_1,\varphi_2}(z_1^1) e_k = -\mathrm{e}^{\mathrm{i}(\varphi_1+\varphi_2)} q^{-1} \sqrt{1 - q^{2k}}\, e_{k-1} \\ \rho_{\varphi_1,\varphi_2}(z_2^1) e_k = \mathrm{e}^{\mathrm{i}\varphi_1} q^k e_k \\ \rho_{\varphi_1,\varphi_2}(z_1^2) e_k = \mathrm{e}^{\mathrm{i}\varphi_2} q^k e_k \\ \rho_{\varphi_1,\varphi_2}(z_2^2) e_k = \sqrt{1 - q^{2(k+1)}}\, e_{k+1} \\ \varphi_i \in [0, 2\pi) \end{array} \tag{6}$$

(4a) infinite-dimensional representations ρ_φ^1 on $H = l_2(\mathbb{Z}^+ \times \mathbb{Z}^+)$

$$\begin{aligned}
&\rho_\varphi^1(z_1^1)e_{m,k} = -\mathrm{e}^{\mathrm{i}\varphi} q^{-1}\sqrt{1-q^{2(m+1)}}\sqrt{1-q^{2k}}e_{m+1,k-1}\\
&\rho_\varphi^1(z_2^1)e_{m,k} = q^k\sqrt{1-q^{2(m+1)}}e_{m+1,k}\\
&\rho_\varphi^1(z_1^2)e_{m,k} = \mathrm{e}^{\mathrm{i}\varphi}q^k e_{m,k}\\
&\rho_\varphi^1(z_2^2)e_{m,k} = \sqrt{1-q^{2(k+1)}}e_{m,k+1}\\
&\varphi \in [0, 2\pi)
\end{aligned} \tag{7}$$

(4b) infinite-dimensional representations ρ_φ^2 on $H = l_2(\mathbb{Z}^+ \times \mathbb{Z}^+)$

$$\begin{aligned}
&\rho_\varphi^2(z_1^1)e_{m,k} = -\mathrm{e}^{\mathrm{i}\varphi} q^{-1}\sqrt{1-q^{2(m+1)}}\sqrt{1-q^{2k}}e_{m+1,k-1}\\
&\rho_\varphi^2(z_2^1)e_{m,k} = \mathrm{e}^{\mathrm{i}\varphi}q^k e_{m,k}\\
&\rho_\varphi^2(z_1^2)e_{m,k} = q^k\sqrt{1-q^{2(m+1)}}e_{m+1,k}\\
&\rho_\varphi^2(z_2^2)e_{m,k} = \sqrt{1-q^{2(k+1)}}e_{m,k+1}\\
&\varphi \in [0, 2\pi)
\end{aligned} \tag{8}$$

(5) infinite-dimensional representations $\hat{\rho}_\varphi$ on $H = l_2(\mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+)$

$$\begin{aligned}
&\rho(z_1^1)e_{m,l,k} = \mathrm{e}^{\mathrm{i}\varphi}q^{m+l}e_{m,l,k} - q^{-1}\sqrt{(1-q^{2(l+1)})(1-q^{2(m+1)})(1-q^{2k})}e_{m+1,l+1,k-1}\\
&\rho(z_2^1)e_{m,l,k} = q^k\sqrt{1-q^{2(m+1)}}e_{m+1,l,k}\\
&\rho(z_1^2)e_{m,l,k} = q^k\sqrt{1-q^{2(l+1)}}e_{m,l+1,k}\\
&\rho(z_2^2)e_{m,l,k} = \sqrt{1-q^{2(k+1)}}e_{m,l,k+1}\\
&\varphi \in [0, 2\pi)
\end{aligned} \tag{9}$$

(6) the infinite-dimensional representation ρ on $H = l_2(\mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+)$

$$\begin{aligned}
\rho(z_1^1)e_{s,m,l,k} &= q^{m+l}\sqrt{1-q^{2(s+1)}}e_{s+1,m,l,k}\\
&\quad -q^{-1}\sqrt{(1-q^{2(l+1)})(1-q^{2(m+1)})(1-q^{2k})}e_{s,m+1,l+1,k-1}\\
\rho(z_2^1)e_{s,m,l,k} &= q^k\sqrt{1-q^{2(m+1)}}e_{s,m+1,l,k}\\
\rho(z_1^2)e_{s,m,l,k} &= q^k\sqrt{1-q^{2(l+1)}}e_{s,m,l+1,k}\\
\rho(z_2^2)e_{s,m,l,k} &= \sqrt{1-q^{2(k+1)}}e_{s,m,l,k+1}.
\end{aligned} \tag{10}$$

Proof. Let us consider a $*$-subalgebra $\mathcal{B}$ of $\mathrm{Pol}(\mathrm{Mat}_{2,2})_q$ which is generated by z_2^1, z_1^2, z_2^2 and $(z_2^1)^*$, $(z_1^2)^*$, $(z_2^2)^*$. Direct computation shows that $z_2^1(z_2^1)^*$, $z_1^2(z_1^2)^*$, $z_2^2(z_2^2)^*$ generate a commutative $*$-subalgebra of $\mathcal{B}$ and satisfy the following relations:

$$(z_a^\alpha(z_a^\alpha)^*)z_b^\beta = z_b^\beta F_{ba}^{\beta\alpha}(z_2^1(z_2^1)^*, z_1^2(z_1^2)^*, z_2^2(z_2^2)^*) \tag{11}$$

where $(\alpha, a), (\beta, b) \in \{(1,2), (2,1), (2,2)\}$ and

$$\begin{aligned}
\mathbb{F}_{21}(x_1,x_2,x_3) &= (F_{22}^{11}(x_1,x_2,x_3), F_{21}^{12}(x_1,x_2,x_3), F_{22}^{12}(x_1,x_2,x_3))\\
&= (q^2x_1 - (1-q^2)(x_3-1), x_2, x_3)\\
\mathbb{F}_{12}(x_1,x_2,x_3) &= (F_{12}^{21}(x_1,x_2,x_3), F_{11}^{22}(x_1,x_2,x_3), F_{12}^{22}(x_1,x_2,x_3))\\
&= (x_1, q^2x_2 - (1-q^2)(x_3-1), x_3)\\
\mathbb{F}_{22}(x_1,x_2,x_3) &= (F_{22}^{21}(x_1,x_2,x_3), F_{21}^{22}(x_1,x_2,x_3), F_{22}^{22}(x_1,x_2,x_3))\\
&= (q^2x_1, q^2x_2, q^2(x_3-1)+1).
\end{aligned}$$

The functions $\mathbb{F}_{21}, \mathbb{F}_{12}, \mathbb{F}_{22} : \mathbb{R}^3 \to \mathbb{R}^3$ define an action of $\mathbb{Z}^3$ on $\mathbb{R}^3$ with orbits

$$\begin{aligned}\Omega_{x_1,x_2,x_3} &= \{\mathbb{F}_{21}^{(m)}(\mathbb{F}_{12}^{(l)}(\mathbb{F}_{22}^{(k)}(x_1, x_2, x_3)))\\ &= (q^{2k}(q^{2m}x_1 - (1 - q^{2m})(x_3 - 1)), q^{2k}(q^{2l}x_2 - (1 - q^{2l})(x_3 - 1))\\ &\quad \times q^{2k}(x_3 - 1) + 1), m, l, k \in \mathbb{Z}\}.\end{aligned}$$

Here and in the following we denote by $\mathbb{F}_{a\alpha}^{(m)}$ the mth iteration of $\mathbb{F}_{a\alpha}$ and $(\mathbb{F}_{a\alpha}^{(m)})_i$, $i = 1, 2, 3$, the ith coordinate of $\mathbb{F}_{a\alpha}^{(m)}$. Let π be a $*$-representation of $\mathrm{Pol}(\mathrm{Mat}_{2,2})_q$ on a Hilbert space H by bounded operators, let $\mathbb{E}(\cdot)$ be the resolution of the identity for the commutative family $\mathbb{A}_\pi$ of the positive operators $\pi(z_2^1)\pi(z_2^1)^*$, $\pi(z_1^2)\pi(z_1^2)^*$, $\pi(z_2^2)\pi(z_2^2)^*$ and let σ_π be the joint spectrum of the family $\mathbb{A}_\pi$.

The next step is to show that any irreducible representation is concentrated on an orbit of this dynamical system.

Lemma 1. *If π is an irreducible representation of* $\mathrm{Pol}(\mathrm{Mat}_{2,2})_q$ *then the spectral measure* $\mathbb{E}(\cdot)$ *is ergodic with respect to the action of the dynamical system generated by* $\mathbb{F}_{21}$, $\mathbb{F}_{12}$, $\mathbb{F}_{22}$ *and there exists an orbit* Ω_{x_1,x_2,x_3} *such that* $\mathbb{E}(\Omega_{x_1,x_2,x_3}) = I$.

Proof. From (11) and the spectral theorem it follows that

$$\mathbb{E}(\Delta)\pi(z_b^\beta) = \pi(z_b^\beta)\mathbb{E}(\mathbb{F}_{b\beta}^{(-1)}(\Delta))$$
$$\mathbb{E}(\Delta)\pi(z_b^\beta)^* = \pi(z_b^\beta)^*\mathbb{E}(\mathbb{F}_{b\beta}(\Delta))$$

for any $\Delta \in \mathfrak{B}(\mathbb{R}^3)$ (Borel sets). Hence any subset Δ such that $\mathbb{F}_{b\beta}^{(-1)}(\Delta) \subseteq \Delta$, $\mathbb{F}_{b\beta}(\Delta) \subseteq \Delta$, $(b, \beta) = (2, 1), (1, 2), (2, 2)$, defines a subspace $\mathbb{E}(\Delta)H$ which is invariant with respect to the operators $\pi(z_b^\beta)$, $\pi(z_b^\beta)^*$ for any (b, β) as above. Moreover, such a subspace is invariant with respect to any operator of the representation π. In fact, the following relations hold in $\mathrm{Pol}(\mathrm{Mat}_{2,2})_q$:

$$z_a^\alpha(z_a^\alpha)^*z_1^1 = z_1^1 z_a^\alpha(z_a^\alpha)^* - (-1)^{a+\alpha}(q - q^{-1})z_2^1 z_1^2(z_2^2)^* \tag{12}$$

$(a, \alpha) = (2, 1), (1, 2), (2, 2)$, which gives

$$\begin{aligned}\mathbb{E}(\mathbb{R}^3 \setminus \Delta)\pi(z_a^\alpha(z_a^\alpha)^*)\pi(z_1^1)\mathbb{E}(\Delta) &= \mathbb{E}(\mathbb{R}^3 \setminus \Delta)\pi(z_1^1)\pi(z_a^\alpha(z_a^\alpha)^*)\mathbb{E}(\Delta)\\ &\quad -(-1)^{a+\alpha}(q - q^{-1})\mathbb{E}(\mathbb{R}^3 \setminus \Delta)\pi(z_2^1)\pi(z_1^2)\pi(z_2^2)^*\mathbb{E}(\Delta).\end{aligned}$$

Therefore if $\Delta \in \mathfrak{B}(\mathbb{R}^3)$ is invariant with respect to all $\mathbb{F}_{b\beta}^{(-1)}$ and $\mathbb{F}_{b\beta}$ we obtain

$$\pi(z_a^\alpha(z_a^\alpha)^*)\mathbb{E}(\mathbb{R}^3 \setminus \Delta)\pi(z_1^1)\mathbb{E}(\Delta) = \mathbb{E}(\mathbb{R}^3 \setminus \Delta)\pi(z_1^1)\mathbb{E}(\Delta)\pi(z_a^\alpha(z_a^\alpha)^*)$$

and hence

$$\mathbb{E}(\Delta')\mathbb{E}(\mathbb{R}^3 \setminus \Delta)\pi(z_1^1)\mathbb{E}(\Delta) = \mathbb{E}(\mathbb{R}^3 \setminus \Delta)\pi(z_1^1)\mathbb{E}(\Delta)\mathbb{E}(\Delta')$$

for any $\Delta' \in \mathfrak{B}(\mathbb{R}^3)$. Taking $\Delta' = \Delta$ gives $\mathbb{E}(\mathbb{R}^3 \setminus \Delta)\pi(z_1^1)\mathbb{E}(\Delta) = 0$, i.e. $\pi(z_1^1)\mathbb{E}(\Delta)H \subseteq \mathbb{E}(\Delta)H$. Similarly, $\pi(z_1^1)^*\mathbb{E}(\Delta)H \subseteq \mathbb{E}(\Delta)H$. The ergodicity of the measure $\mathbb{E}(\cdot)$ follows immediately; i.e., $\mathbb{E}(\Delta) = I$ or 0 for any Borel Δ which is invariant with respect to $\mathbb{F}_{b\beta}$, $\mathbb{F}_{b\beta}^{(-1)}$.

The simplest invariant sets are the orbits of the dynamical system. The next step is to show that only atomic measures concentrated on an orbit give rise to an irreducible representation of the $*$-algebra. It is easily seen that the dynamical system generated by $\mathbb{F}_{b\beta}$ is one to one and possesses a measurable section, i.e. a set $\tau \in \mathfrak{B}(\mathbb{R}^3)$ which intersects any orbit in a single point. This implies that any ergodic measure is concentrated on a single orbit of the dynamical system and therefore $\mathbb{E}(\Omega_{x_1,x_2,x_3}) = I$ for some orbit Ω_{x_1,x_2,x_3}. □

We now clarify which orbits Ω_{x_1,x_2,x_3} give rise to bounded irreducible representations π, i.e. $\sigma_\pi \subseteq \Omega_{x_1,x_2,x_3}$, and classify all such representations up to unitary equivalence.

We claim first that there is no bounded representation π with $\sigma_\pi \subseteq \Omega_{x_1,x_2,x_3}$ if $x_3 > 1$. From (11) we have

$$\pi(z_b^\beta)H_x \subseteq H_{\mathbb{F}_{b\beta}(x)} \qquad \pi(z_b^\beta)^* H_x \subseteq H_{\mathbb{F}_{b\beta}^{(-1)}(x)} \tag{13}$$

where H_x is the eigenspace for $\mathbb{A}_\pi$ corresponding to the eigenvalue $x \in \mathbb{R}^3$. Since $y = (y_1, y_2, y_3) \in \Omega_{x_1,x_2,x_3}$, where $x_3 > 1$, which implies that $y_3 > 1$ we conclude that $\pi(z_2^2)\pi(z_2^2)^* \geqslant 1$ and $\ker \pi(z_2^2) = \ker \pi(z_2^2)^* = \{0\}$. This clearly forces $\mathbb{F}_{22}^{(k)}(y) \in \sigma_\pi$ for any $k \in \mathbb{Z}$. However, the set $\{\mathbb{F}_{22}^{(k)}(y), k \in \mathbb{Z}\}$ is unbounded, which contradicts the boundedness of the representation π. Similar arguments show that there is no bounded representation π with $\sigma_\pi \subseteq \Omega_{x_1,x_2,1}, x_1 \neq 0$ or $x_2 \neq 0$. In this case $\Omega_{x_1,x_2,1} = \{(q^{2(k+m)}x_1, q^{2(k+l)}x_2, 1), k, l, m \in \mathbb{Z}\}$. The only possibility is $\sigma_\pi = \Omega_{0,0,1} = \{(0, 0, 1)\}$ and in this case we obtain $\pi(z_2^1) = \pi(z_1^2) = 0$, $\pi(z_2^2)\pi(z_2^2)^* = I$. It follows now from (1)–(3) that $\pi(z_2^2)$, $\pi(z_1^1)$ satisfy the relations

$$\begin{aligned}
&\pi(z_1^1)^*\pi(z_1^1) = q^2\pi(z_1^1)\pi(z_1^1)^* + (q^{-2} - 1)\\
&[\pi(z_1^1), \pi(z_1^2)] = 0 \qquad [\pi(z_1^1)^*, \pi(z_1^2)] = 0\\
&\pi(z_2^2)^*\pi(z_2^2) = \pi(z_2^2)\pi(z_2^2)^* = I.
\end{aligned} \tag{14}$$

This implies that $\pi(z_2^2)$ commutes with all images of the generators in the algebra under the representation π and therefore $\pi(z_2^2)$ is a multiple of the identity operator if π is irreducible. By (14) we have also $\pi(z_2^2) = \mathrm{e}^{\mathrm{i}\varphi_2} I$, $\varphi_2 \in [0, 2\pi)$. Irreducible representations of the relation $(z_1^1)^* z_1^1 = q^2 z_1^1 (z_1^1)^* + (q^{-2} - 1)$ are well known and can be easily calculated using the method of dynamical systems (see [OS, Chapter 2]). Any such representation is either one dimensional, $\xi_{\varphi_1}(z_1^1) = q^{-1}\mathrm{e}^{\mathrm{i}\varphi_1}$, $\varphi_1 \in [0, 2\pi)$, or infinite dimensional, which is unitarily equivalent to the following one: $\pi_\varphi(z_1^1)e_k = q^{-1}\sqrt{1 - q^{2(k+1)}}e_{k+1}$. The corresponding irreducible representations of $\mathrm{Pol}(\mathrm{Mat}_{2,2})_q$ are $\xi_{\varphi_1,\varphi_2}$ and π_φ.

Since $\sigma_\pi \subseteq (\mathbb{R}^+)^3$ and $(\mathbb{F}_{22}^{(k)})_3(x_1, x_2, x_3) = q^{2k}(x_3 - 1) + 1 \to -\infty$ as $k \to -\infty$, it follows from (13) that $\ker \pi(z_2^2)^* \neq \{0\}$, $\ker \pi(z_2^2)\pi(z_2^2)^* \neq \{0\}$ and the corresponding orbit contains a point $(x_1, x_2, 0)$. We have $\Omega_{x_1,x_2,0} = \{(q^{2k}(q^{2m}(x_1 - 1) + 1), q^{2k}(q^{2l}(x_2 - 1) + 1), 1 - q^{2k}), m, l, k \in \mathbb{Z}\}$. Similar arguments show that $\sigma_\pi \subseteq \Omega_{x_1,x_2,0}$, where $x_1 > 1$ or $x_2 > 1$, is impossible if the representation π is bounded. From the positiveness of σ_π we also obtain that the only orbits corresponding to the irreducible representation of the $*$-algebra are $\Omega_{1,1,0}$, $\Omega_{1,0,0}$, $\Omega_{0,1,0}$, $\Omega_{0,0,0}$ and $\Omega_{0,0,1}$. The last one was treated above.

We consider now the case $\sigma_\pi \subseteq \Omega_{x_1,x_2,x_3}$, $x_3 = 0$. Let P_y, $y \in \mathbb{R}$, be the projection onto the eigenspace corresponding to the eigenvalue y. Using (12) we get

$$(z_k - y_k)P_z\pi(z_1^1)P_y = \pm(q - q^{-1})P_z\pi(z_2^1)\pi(z_1^2)\pi(z_2^2)^*P_y$$

('+' for $k = 1, 2$ and '−' for $k = 3$) for $z, y \in \mathbb{R}^3$. By (13) we have $\pi(z_2^1)\pi(z_1^2)\pi(z_2^2)^*H_y \subseteq H_{\mathbb{F}_{21}(\mathbb{F}_{12}(\mathbb{F}_{22}^{(-1)}(y)))}$ and

$$\pi(z_2^1)\pi(z_1^2)\pi(z_2^2)^*P_y = P_{\mathbb{F}_{21}(\mathbb{F}_{12}(\mathbb{F}_{22}^{(-1)}(y)))}\pi(z_2^1)\pi(z_1^2)\pi(z_2^2)^*P_y.$$

Setting $P_{m,l,k}$ as the projection onto an eigenspace which corresponds to the eigenvalue $\mathbb{F}_{21}^{(m)}(\mathbb{F}_{12}^{(l)}(\mathbb{F}_{22}^{(k)}(x_1, x_2, 0)))$ we obtain

$$\pi(z_1^1)P_{m,l,k} = P_{m,l,k}\pi(z_1^1)P_{m,l,k} + P_{m+1,l+1,k-1}\pi(z_1^1)P_{m,l,k}$$

i.e.

$$\pi(z_1^1)H_{m,l,k} \subseteq H_{m,l,k} \oplus H_{m+1,l+1,k-1}.$$

Moreover, $P_{m+1,l+1,k-1}\pi(z_1^1)P_{m,l,k} = -q^{1-2k}\pi(z_2^1)\pi(z_1^2)\pi(z_2^2)^*P_{m,l,k}$. The operator $\pi(z_1^1)$ can be written as a sum of its diagonal part $\pi(z_1^1)_0 = \sum_{m,l,k} P_{m,l,k}\pi(z_1^1)P_{m,l,k}$, and the operator $-\sum_{m,l,k} q^{1-2k}\pi(z_2^1)\pi(z_1^2)\pi(z_2^2)^*P_{m,l,k} = -q\pi(z_2^1)\pi(z_1^2)\pi(z_2^2)^*(1 - \pi(z_2^2(z_2^2)^*))^{-1}$.

Now let $\sigma_\pi \subseteq \Omega_{x_1,x_2,0}$, where $x_1 \neq 0$ or $x_2 \neq 0$. It follows from (1)–(3) by direct computation that

$$\pi(z_1^1)_0^*\pi(z_1^1)_0 = q^2\pi(z_1^1)_0\pi(z_1^1)_0^*.$$

The only bounded operator $\pi(z_1^1)_0$ satisfying this relation is the zero operator. Therefore

$$\pi(z_1^1) = -q\pi(z_2^1)\pi(z_1^2)\pi(z_2^2)^*(1-\pi(z_2^2(z_2^2)^*))^{-1}$$

and π is irreducible iff so is the family $(\pi(z_2^1), \pi(z_1^2), \pi(z_2^2), \pi(z_2^1)^*, \pi(z_1^2)^*, \pi(z_2^2)^*)$. Let $\pi(z_a^\alpha) = U_a^\alpha\sqrt{\pi(z_a^\alpha)^*\pi(z_a^\alpha)}$ be the polar decomposition of $\pi(z_a^\alpha)$. Using simple arguments one can show that $[U_a^\alpha, U_b^\beta] = [(U_a^\alpha)^*, U_b^\beta] = 0$, $(a,\alpha) \neq (b,\beta)$ and

$$(z_a^\alpha(z_a^\alpha)^*)(U_b^\beta) = (U_b^\beta)F_{ba}^{\beta\alpha}(z_2^1(z_2^1)^*, z_1^2(z_1^2)^*, z_2^2(z_2^2)^*).$$

Here $(a,\alpha), (b,\beta) \in \{(1,2),(2,1),(2,2)\}$. Moreover, if $\sigma_\pi \subseteq \Omega_{1,x_2,0}$ ($\sigma_\pi \subseteq \Omega_{x_1,1,0}$) we have U_2^1 (U_1^2 respectively) commutes with any operators from the family $\mathbb{A}_\pi$ and therefore with any operator of the representation. This clearly forces $U_2^1 = \mathrm{e}^{\mathrm{i}\varphi_1}I$, $\varphi_1 \in [0,2\pi)$ ($U_1^2 = \mathrm{e}^{\mathrm{i}\varphi_2}I$, $\varphi_2 \in [0,2\pi)$ respectively). Let $\sigma_\pi \subseteq \Omega_{0,1,0}$. Consider $e_{k,l} = (U_2^1)^k(U_2^2)^l e$, $e \in \ker\pi(z_2)\pi(z_2^2)^* \cap \ker\pi(z_2^1)\pi(z_2^1)^*$, $k,l \in \mathbb{Z}^+$. Then $\{e_{k,l}, k,l \in \mathbb{Z}^+\}$ is an orthonormal system which defines an invariant subspace. The corresponding irreducible representation is ρ_φ^2. Analogously $(U_1^2)^k(U_2^2)^l e = e_{k,l}$, $e \in \ker\pi(z_2)\pi(z_2^2)^* \cap \ker\pi(z_1^2)\pi(z_1^2)^*$, $k,l \in \mathbb{Z}^+$, build an orthonormal basis of an irreducible representation space if $\sigma_\pi \subseteq \Omega_{1,0,0}$; the corresponding action is given by formulae (8). If $\sigma_\pi \subseteq \Omega_{1,1,0}$ we have that $l.s.\{(U_2^2)^k e = e_k, k \in \mathbb{Z}^+\}$, $e \in \ker\pi(z_2^2)\pi(z_2^2)^*$, is invariant with the corresponding action given by (6).

We now turn to the case $\sigma_\pi \subset \Omega_{0,0,0}$. From (1)–(3) we have

$$\begin{aligned}
&\pi(z_1^1)_0\pi(z_1^2) = q\pi(z_1^2)\pi(z_1^1)_0 && \pi(z_1^1)_0^*\pi(z_1^2) = q\pi(z_1^2)\pi(z_1^1)_0^* \\
&\pi(z_1^1)_0\pi(z_2^1) = q\pi(z_2^1)\pi(z_1^1)_0 && \pi(z_1^1)_0^*\pi(z_2^1) = q\pi(z_2^1)\pi(z_1^1)_0^* \\
&\pi(z_1^1)_0\pi(z_2^2) = \pi(z_2^2)\pi(z_1^1)_0 && \pi(z_1^1)_0^*\pi(z_2^2) = \pi(z_2^2)\pi(z_1^1)_0^* \\
&\pi(z_1^1)_0^*\pi(z_1^1)_0 P_{m,l,k} = q^2\pi(z_1^1)_0\pi(z_1^1)_0^* P_{m,l,k} + (1-q^2)q^{2(m+l)}P_{m,l,k}.
\end{aligned}$$

Note that $\pi(z_1^1)_0 P_{m,l,k}H \subseteq P_{m,l,k}H$, $\pi(z_1^1)_0^* P_{m,l,k}H \subseteq P_{m,l,k}H$. Moreover, it follows from the above relation that if π is irreducible then the family $(\pi(z_1^1)_0, \pi(z_1^1)_0^*)$ restricted to the subspace $P_{m,l,k}H$ is irreducible for any $m,l,k \in \mathbb{Z}^+$. We have

$$a^*a = q^2aa^* + (1-q^2)$$

where $a = \pi(z_1^1)_0 P_{0,0,0}$. Any irreducible family (a, a^*) is either one dimensional and given by $a = \mathrm{e}^{\mathrm{i}\varphi}$, $\varphi \in [0,2\pi)$, or infinite dimensional defined on $l_2(\mathbb{Z}^+)$ by $ae_s = \sqrt{1-q^{2(s+1)}}e_{s+1}$. These representations give rise to irreducible representations of the $*$-algebra $\mathrm{Pol}(\mathrm{Mat}_{2,2})_q$. Namely, in the first case we have that $e_{m,l,k} = (U_2^1)^m(U_1^2)^l(U_2^2)^k e$, where $e \in P_{0,0,0}H = \ker\pi(z_2^2)\pi(z_2^2)^* \cap \ker\pi(z_1^2)\pi(z_1^2)^* \cap \ker\pi(z_2^1)\pi(z_2^1)^*$, $m,l,k \in \mathbb{Z}^+$, define an orthonormal basis of the space where the irreducible representation $\hat{\rho}_\varphi$ acts, and for the second irreducible family we have that $e_{s,m,l,k} = (U_2^1)^m(U_1^2)^l(U_2^2)^k e_s$, $s,m,l,k \in \mathbb{Z}^+$, define an orthonormal basis of the space where the irreducible representation ρ acts. This completes the proof. □

Comments. It follows from the proof that for any representation π on a Hilbert space H_π the family of self-adjoint operators $\pi(z_2^2(z_2^2)^*)$, $\pi(z_2^1(z_2^1)^*)$, $\pi(z_1^2(z_1^2)^*)$, $\pi(z_1^1)_0\pi(z_1^1)_0^*$, where

$$\pi(z_1^1)_0 = \pi(z_1^1) - \begin{cases} 0 & \pi(z_2^2(z_2^2)^*) = I \\ -q\pi(z_2^1)\pi(z_1^2)\pi(z_2^2)^*(1-\pi(z_2^2(z_2^2)^*))^{-1} & \pi(z_2^2(z_2^2)^*) \neq I \end{cases}$$

generates a commutative $*$-subalgebra $\mathbb{A}$ in $B(H_\pi)$, the bounded linear operators on H_π. Moreover, any irreducible representation of $\mathrm{Pol}(\mathrm{Mat}_{2,2})_q$ is a weight representation with respect to this algebra, i.e. $\mathbb{A}$ can be diagonalized, and the spectrum of $\mathbb{A}$ is simple.

A question which arises here is how to generalize the method to higher-dimension matrix balls and classify $*$-representations of the corresponding $*$-algebras. In principle, just by analysing the commutation relations between the generators in the $*$-algebra one can find a commutative $*$-subalgebra of $\mathrm{Pol}(\mathrm{Mat}_{m,n})_q$ or one of its localizations and show that any irreducible representation π is a weight representation with respect to this commutative $*$-algebra having a simple spectrum in this representation. However, the computations can be extremely difficult in general.

Remark 1. The polynomial algebra on the vector space $\mathrm{Mat}_{2,2}$ can be supplied with a Poisson structure. Writing $q = \mathrm{e}^{-h}$ we have that $\mathrm{Pol}(\mathrm{Mat}_{2,2})_{\exp(-h)}$ is an associative algebra over the ring of formal series $\mathbb{C}[[h]]$ and

$$\mathrm{Pol}(\mathrm{Mat}_{2,2}) \simeq \mathrm{Pol}(\mathrm{Mat}_{2,2})_{\exp(-h)}/h\mathrm{Pol}(\mathrm{Mat}_{2,2})_{\exp(-h)}.$$

The Poisson bracket is now given by

$$\{a \bmod h, b \bmod h\} = -\mathrm{i}h^{-1}(ab - ba) \bmod h$$

for any $a, b \in \mathrm{Pol}(\mathrm{Mat}_{2,2})_{\exp(-h)}$. The problem now is to define the symplectic leaves of this Poisson structure. Any primitive ideal $\ker\pi$, where π is an irreducible representation of $\mathrm{Pol}(\mathrm{Mat}_{2,2})_q$, defines a maximal Poisson ideal $I_\pi = \ker\pi \bmod h$ of the algebra $\mathrm{Pol}(\mathrm{Mat}_{2,2})$ ordered by inclusion and hence the closure of a symplectic leaf which is given by $\{x \in \mathrm{Mat}_{2,2} \mid f(x) = 0,\ f \in I_\pi\}$. As in the case of $\mathbb{C}(SU(n))_q$ (see [SoV]) one can expect that there is a one-to-one correspondence between irreducible representations (bounded irreducible representations) of $\mathrm{Pol}(\mathrm{Mat}_{2,2})_q$ and symplectic leaves (bounded symplectic leaves) in $\mathrm{Mat}_{2,2}$.

3. Representations of $\mathrm{Pol}(\mathrm{Mat}_{2,2})_q$ and the Shilov boundary of the matrix ball

It is known that the Shilov boundary $S(\mathbb{U})$ of the matrix ball $\mathbb{U} = \{z \in \mathrm{Mat}_{m,m} \mid z^*z < 1\}$ is the set of all unitary $m \times m$ matrices. A q-analogue $\mathrm{Pol}(S(\mathbb{U}))_q$ of the polynomial algebra on the Shilov boundary of $\mathbb{U}$ was introduced in [V] and shown to be isomorphic to $\mathbb{C}(U_n)_q = (\mathbb{C}(Gl_n)_q, *)$, the algebra of regular functions on the quantum group U_n (see, e.g., [ChP]). We recall that $\mathbb{C}(Gl_2)_q$ is the localization of the algebra $\mathbb{C}(\mathrm{Mat}_{2,2})_q$ with respect to the multiplicative system $(\det_q z)^{\mathbb{N}}$, where $\det_q z$ is the quantum determinant $z_1^1 z_2^2 - q z_2^1 z_1^2$. For $m = 2$ the $*$-algebra $\mathrm{Pol}(S(\mathbb{U}))_q$ is equal to $(\mathbb{C}(Gl_2)_q, *)$, where the involution $*$ is as follows

$$\begin{pmatrix} (z_1^1)^* & (z_2^1)^* \\ (z_1^2)^* & (z_2^2)^* \end{pmatrix} = (z_1^1 z_2^2 - q z_2^1 z_1^2)^{-1} \begin{pmatrix} q^{-2} z_2^2 & -q^{-1} z_1^2 \\ -q^{-1} z_2^1 & z_1^1 \end{pmatrix}.$$

We define $\psi : \mathrm{Pol}(\mathrm{Mat}_{2,2}) \mapsto \mathrm{Pol}(S(\mathbb{U}))_q$ by setting $\psi(z_a^\alpha) = z_a^\alpha$. By [V][theorem 2.2], ψ can be uniquely extended to a $*$-homomorphism of these $*$-algebras. Therefore, ψ is considered as a q-analogue of the restriction operator of polynomials to the Shilov boundary. The isomorphism $\mathrm{Pol}(S(\mathbb{U}))_q \simeq \mathbb{C}(U_2)_q$ is given by $\tau : z_a^\alpha \mapsto q^{\alpha-2} z_a^\alpha$, $a, \alpha = 1, 2$. It is clear that any representation π of $\mathbb{C}(U_2)_q$ generates a representation $\pi' = \pi \circ \tau \circ \psi$ of the algebra $\mathrm{Pol}(\mathrm{Mat}_{2,2})_q$. We have the following proposition.

Proposition 1. *The only irreducible representations of* $\mathrm{Pol}(\mathrm{Mat}_{2,2})_q$ *which are induced by a representation of* $\mathbb{C}(U_2)_q$ *are* $\xi_{\varphi_1,\varphi_2}$ *and* $\rho_{\varphi_1,\varphi_2}$ *for any* $\varphi_1, \varphi_2 \in [0, 2\pi)$.

Proof. Follows by direct verification. □

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